

Closing Thurs: HW 9.4

Closing Tues: HW 9.5

HW Help: MSC 12:30-4:30 Mon-Thu

Recall: Power, Sum, Coeff. Rules

$$\frac{d}{dx}(x^n) = n x^{n-1}.$$

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$

$$\frac{d}{dx}(cf(x)) = cf'(x).$$

- Steps:**
1. Expand.
 2. Rewrite Powers.
 3. Use power rule.

Entry Task: Differentiate

$$A) y = 5x^3 - \frac{x^{10}}{2}$$

$$B) y = x\sqrt{x} - 6 + \frac{3}{\sqrt{x}} = x^1 x^{1/2} - 6 + 3x^{-1/2}$$

$$C) y = 5(2x)^3 = 5 \cdot 8 \cdot x^3$$

$$D) y = 2 \left(\frac{x^2}{3}\right) \left(\frac{9}{x^4}\right) x^3 = 2 \cdot \frac{9}{3} \cdot \frac{x^2 x^3}{x^4} = 6x$$

$$\boxed{A} \quad y = 5x^3 - \frac{1}{2}x^{10} \Rightarrow \boxed{\frac{dy}{dx} = 15x^2 - 5x^9}$$

$$\boxed{B} \quad y = x^{3/2} - 6 + 3x^{-1/2} \Rightarrow \boxed{\frac{dy}{dx} = \frac{3}{2}x^{1/2} - 0 - \frac{3}{2}x^{-3/2}}$$

$$\boxed{C} \quad y = 40x^3 \Rightarrow \boxed{\frac{dy}{dx} = 120x^2}$$

$$\boxed{D} \quad y = 6x \Rightarrow \boxed{\frac{dy}{dx} = 6}$$

5: Marginal Analysis and the Product/Quotient Rules

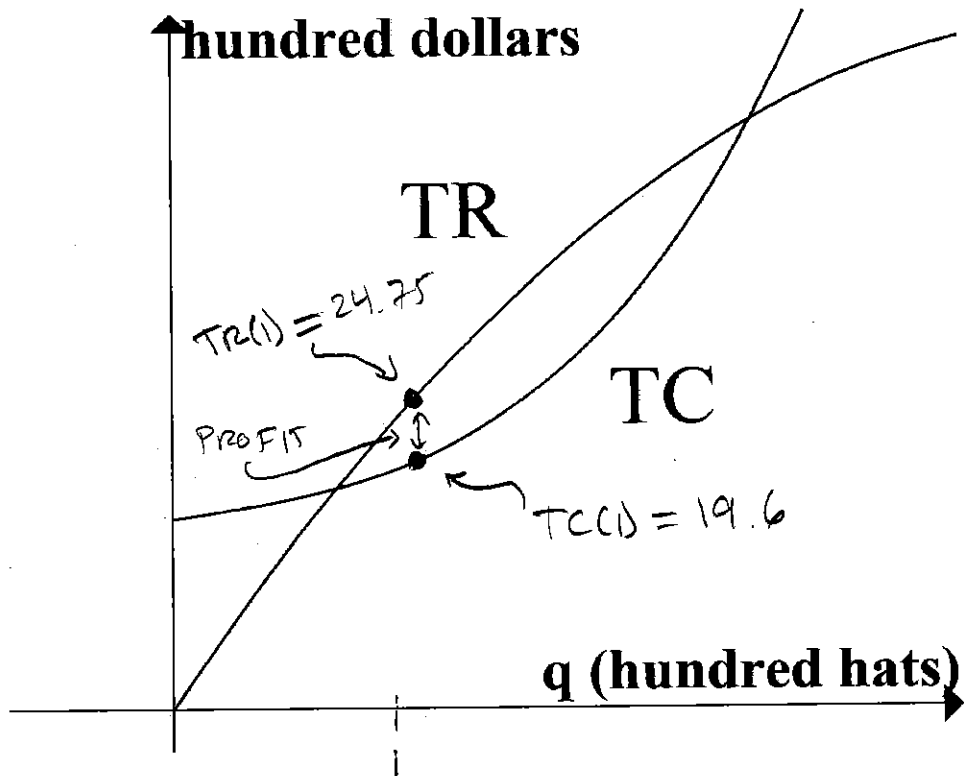
Example: Your company produces and sells hats. Based on data from recent months, you estimate:

Total Revenue:

$$TR(q) = -3.75q^2 + 28.5q$$

Total Cost:

$$TC(q) = 2q^3 - 0.4q^2 + 3q + 15$$



These functions give revenue and cost in a month where you produce q hundred hats

$TR(q), TC(q)$ are in hundred dollars

Quick Review Question:

What is $TR(0)$? What is $TC(0)$?

What do these represent?

$$TR(0) = -3.75(0)^2 + 28.5(0) = 0$$

$$TC(0) = 15 \text{ hundred dollars}$$

If you sell 0 hundred hats
your revenue is 0 hundred dollars.

If you produce 0 hundred hats
your cost is 15 hundred dollars
(1500)

Numerical Review:

When $q = 1$ hundred hats (100 hats):

$$\begin{aligned} TR(1) &= -3.75(1)^2 + 28.5(1) \\ &= 24.75 \quad \text{hundred dollars} \end{aligned}$$

$$\begin{aligned} TC(1) &= 2(1)^3 - 0.4(1)^2 + 3(1) + 15 \\ &= 19.6 \quad \text{hundred dollars} \end{aligned}$$

$$\begin{aligned} \text{Profit} = P(1) &= TR(1) - TC(1) \\ &= 24.75 - 19.6 \\ &= 5.15 \text{ hundred dollars} \end{aligned}$$

Thus, if you produce and sell exactly 100 hats this month, then your profit will be \$515.

$$\begin{aligned} 100 \text{ hats} &\Rightarrow TR \text{ of } \$2475 \\ 100 \text{ hats} &\Rightarrow TC \text{ of } \$1960 \\ \hline &\text{Profit } \$515 \end{aligned}$$

new Definition

In Math 112, we are going to redefine Marginal Revenue and Marginal Cost as follows:

$$MR(q) = TR'(q)$$

$$MC(q) = TC'(q)$$

Q: Using this new definition and our derivative rules, find $MR(q)$ and $MC(q)$.

$$TR(q) = -3.75q^2 + 28.5q$$

$$MR(q) = -3.75 \cdot 2q + 28.5$$

$$MR(q) = -7.5q + 28.5$$

$$TC(q) = 2q^3 - 0.4q^2 + 3q + 15$$

$$MC(q) = 6q^2 - 1.2q + 3$$

Q: What is $MR(2)$?

What is $MC(2)$?

What are the units?

$$MR(2) = -7.5(2) + 28.5 = 13.5$$

$$MC(2) = 6(2)^2 - 0.8(2) + 3 = 25.4$$

→ slope units = $\frac{\text{hundred dollars}}{\text{hundred hats}}$
 $= \frac{\text{dollars}}{\text{hat}}$

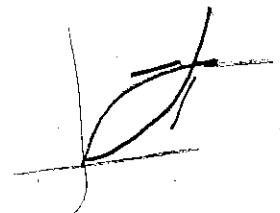
$$MR(2) = \$13.50 \frac{\text{dollars}}{\text{hat}}$$

$$MC(2) = \$25.40 \frac{\text{dollars}}{\text{hat}}$$

⇒ TC is "steeper" than TR at 2 hundred hats.

⇒ PROFIT DECREASING

WOULD YOU RATHER
PRODUCE/SELL
200 or 201?



Comparing the Math 111 and 112 Definitions of MR and MC

Math 111:

$$\begin{aligned}MR(2) &= \text{change in revenue you go} \\ &\quad \text{from selling 200 to 201 hats} \\ &= TR(2.01) - TR(2) \\ &= 0.134625 \text{ hundred dollars} \\ &\quad (13.46 \text{ dollars})\end{aligned}$$

$$\begin{aligned}MC(2) &= \text{change in cost you go} \\ &\quad \text{from making 200 to 201 hats} \\ &= TC(2.01) - TC(2) \\ &= 0.255162 \text{ hundred dollars} \\ &\quad (25.52 \text{ dollars})\end{aligned}$$

Math 112:

$$MR(2) = TR'(2) = 13.50 \text{ dollars/hat}$$

$$MC(2) = TC'(2) = 25.40 \text{ dollars/hat}$$

NOTES:

1. The Math 111 and Math 112 definitions for MR and MC differ by 5 and 12 cents, respectively. (they are *close* to the same).
2. The derivative definition is much easier to compute and use.
3. We use the Math 111 definition to interpret and analyze our results.

NOTE IN GENERAL, THE DERIVATIVE
IS APPROXIMATELY EQUAL
TO THE CHANGE OVER THE
"NEXT ITEM".

From HW: $q = \frac{108}{\sqrt{p}} - 1$

q = UNITS

p = PRICE

$$q = 108 p^{-1/2} - 1$$

$$q'(p) = -54 p^{-3/2}$$

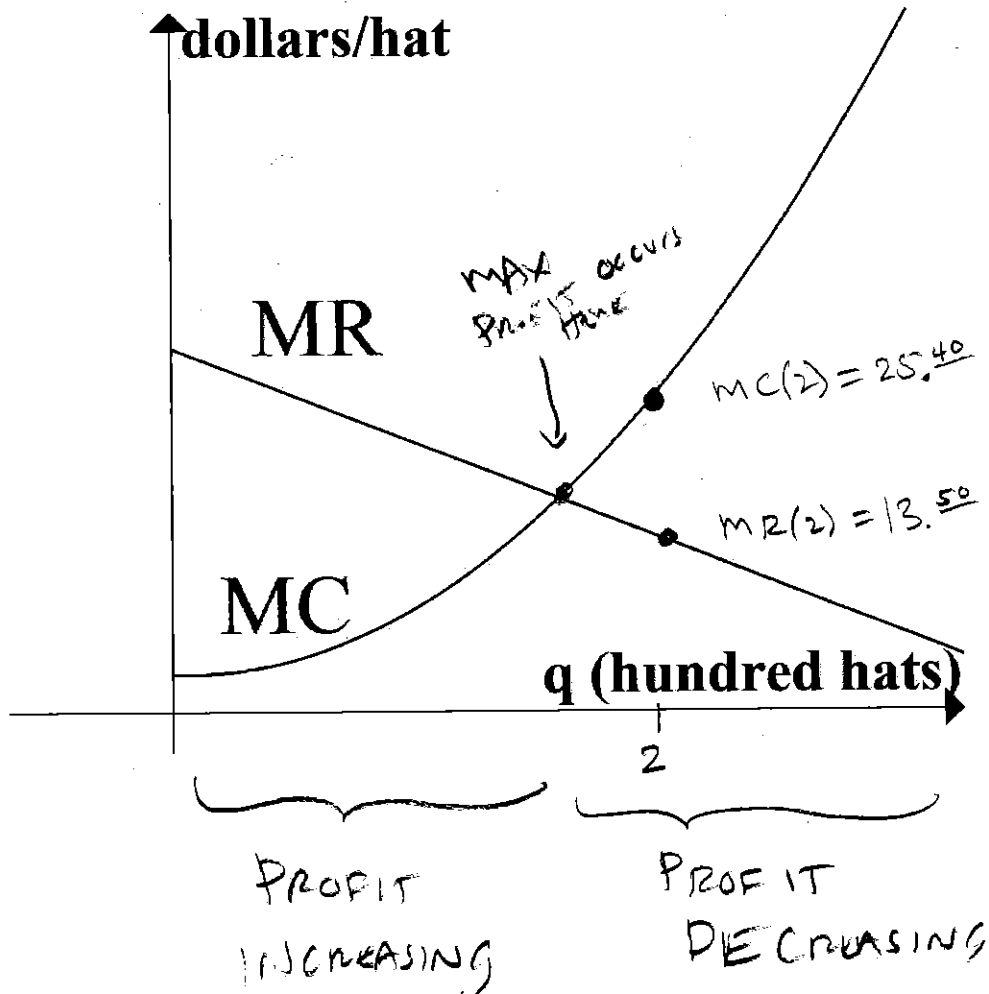
≈ CHANGE IN
UNITS IF
PRICE IS INCREASED
BY ONE.

Δ Marginal Revenue:

$$MR(q) = -7.5q + 28.5$$

Δ Marginal Cost:

$$MC(q) = 6q^2 - 0.8q + 3$$



Marginal Analysis

recall a big observation

from Math 111:

• $MR(q) > MC(q)$, then profit is increasing at q .

(Revenue has a higher slope than cost)

• $MR(q) < MC(q)$, then profit is decreasing at q .

(Revenue has a lower slope than cost)

Thus, profit is maximized at the quantity when it switches from $MR(q) > MC(q)$ to $MR(q) < MC(q)$

In other words, **profit is maximized where $MR(q) = MC(q)$.**

(match slopes)

our example:

$$TR(q) = -3.75q^2 + 28.5q$$

$$TC(q) = 2q^3 - 0.4q^2 + 3q + 15$$

o maximize profit:

step 1: Find $MR(q)$ and $MC(q)$:

$$MR(q) = TR'(q) = -7.5q + 28.5 \quad \checkmark$$

$$MC(q) = TC'(q) = 6q^2 - 0.8q + 3 \quad \checkmark$$

step 2: Solve $MR(q) = MC(q)$

$$-7.5q + 28.5 \stackrel{?}{=} 6q^2 - 0.8q + 3$$

$+7.5q - 28.5$

$$\Rightarrow 0 = 6q^2 + 6.7q - 25.5$$

$\begin{matrix} \uparrow & & \uparrow & & \uparrow \\ a & & b & & c \end{matrix}$

$$q = \frac{-6.7 \pm \sqrt{6.7^2 - 4(6)(-25.5)}}{2(6)} \approx$$

-2.69

or

1.58

Product 1500
158 hats this
month to maximize
profit

hundred
hats

three more derivative rules.

$$\text{PRODUCT RULE: } \frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x).$$

$$(F \cdot S)' = FS' + F'S$$

Ex) $y = \underbrace{(x^2 + 5x)}_F \underbrace{(x^4 - x^3 + 1)}_S$

$$y' = \underbrace{(x^2 + 5x)}_F \underbrace{(4x^3 - 3x^2)}_{S'} + \underbrace{(2x + 5)}_{F'} \underbrace{(x^4 - x^3 + 1)}_S$$

Ex) $y = \underbrace{x^3}_F \underbrace{(x^2 + x^{10})}_S \longrightarrow \text{OR EXPAND FIRST, } y = x^5 + x^{13}$

$$y' = \underbrace{x^3}_F \underbrace{(2x + 10x^9)}_{S'} + \underbrace{3x^2}_{F'} \underbrace{(x^2 + x^{10})}_S$$

$2x^4 + 10x^{12} + 3x^4 + 3x^{12} = 5x^4 + 13x^{12}$ \leftarrow SAME \downarrow CHECK \nearrow YES!

QUOTIENT RULE:

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\left(\frac{N}{D} \right)' = \frac{DN' - ND'}{D^2}$$

Ex) $y = \frac{9x^2 - x}{14x + 7}$ $\leftarrow N$ $\leftarrow D$ $\frac{dy}{dx} = \frac{(14x+7)(18x-1) - (9x^2-x)(14)}{(14x+7)^2}$

Ex) $y = \frac{4x+2}{x^3}$

or

$$y = \frac{4x}{x^3} + \frac{2}{x^3} = 4x^{-2} + 2x^{-3}$$

$$y' = -8x^{-3} - 6x^{-4}$$

$$y' = \frac{x^3(4) - (4x+2)3x^2}{x^6}$$

\leftarrow SAME!

\hookrightarrow CHECK

$$\frac{4x^3 - 12x^3 - 6x^2}{x^6} = \frac{-8x^3 - 6x^2}{x^6} = -8x^{-3} - 6x^{-4}$$

ou try: Differentiate

$$y = x^2(x^3 + 1)$$

EXPANDING $\Rightarrow y = x^5 + x^2$

$$\frac{dy}{dx} = 5x^4 + 2x$$

OR

PRODUCT RULE $\Rightarrow \frac{dy}{dx} = \underbrace{x^2}_F \cdot \underbrace{3x^2}_{S'} + \underbrace{2x}_{F'} \cdot \underbrace{(x^3+1)}_S$

$$= 3x^4 + 2x^4 + 2x$$

$$= \boxed{5x^4 + 2x}$$

$$\therefore y = \frac{5}{x^3}$$

REWRITING $\Rightarrow y = 5x^{-3}$

$$\frac{dy}{dx} = -15x^{-4}$$

OR

QUOTIENT RULE $\frac{dy}{dx} = \frac{x^3 \cdot (0) - 5 \cdot 3x^2}{x^6}$
 $= \frac{-15x^2}{x^6} = \boxed{-15x^{-4}}$

$$3. y = (x^2 + 3x)(\sqrt{x} - 5x^3)$$

EXPANDING $\Rightarrow y = x^{2.5} - 5x^5 + 3x^{1.5} - 15x^4$

$$y' = 2.5x^{1.5} - 25x^4 + 4.5x^{0.5} - 60x^3$$

OR

PRODUCT RULE $\Rightarrow y' = (x^2 + 3x) \left(\frac{1}{2}x^{-1/2} - 15x^2 \right) + (2x + 3)(x^{1/2} - 5x^3)$

$$= \frac{1}{2}x^{1.5} - 15x^4 + \frac{3}{2}x^2 - 45x^3 + 2x^{1.5} - 10x^4 + 3x^{0.5} - 15x^3$$

$$= \boxed{2.5x^{1.5} - 25x^4 + 4.5x^{0.5} - 60x^3}$$

$$4. y = \frac{x^5}{3x^3 - x^5}$$

QUOTIENT RULE

$$\Rightarrow y' = \frac{(3x^3 - x^5)5x^4 - x^5(9x^2 - 5x^4)}{(3x^3 - x^5)^2}$$

$$= \frac{3x^7 - 5x^9 - 9x^7 + 5x^9}{(3x^3 - x^5)^2}$$

$$= \frac{-6x^7}{(3x^3 - x^5)^2}$$

CHAIN

~~CHAIN~~ RULE:

$$\frac{d}{dx}(f(g(x))) = f'(g(x))f'(x)$$

EX) $y = (3x^2 + 1)^4$

OUTSIDE = $(\quad)^4 \Rightarrow f(u) = u^4$
INSIDE = $3x^2 + 1 \Rightarrow g(x) = 3x^2 + 1$ } $f(g(x)) = (3x^2 + 1)^4$

$$u^4 \rightarrow 4u^3$$

$$3x^2 + 1 \rightarrow 6x$$

LEAVE INSIDE HERE

$$y' = 4u^3 \cdot 6x = \underbrace{4(3x^2 + 1)^3}_{\text{DERIV. OF "OUTSIDE"}} \cdot \underbrace{6x}_{\text{DERIV. OF "INSIDE"}}$$

EX) $y = (6x + x^{14})^{100}$

$$y' = 100(6x + x^{14})^{99} \cdot (6 + 14x^{13})$$